

# Fault Currents, Circuit Breakers, and a New Method for X/R Calculations in Parallel Circuits

Amir Norouzi  
*Schweitzer Engineering Laboratories, Inc.*

Presented at the  
22nd Annual Georgia Tech Fault and Disturbance Analysis Conference  
Atlanta, Georgia  
April 29–30, 2019

Originally presented at the  
72nd Annual Conference for Protective Relay Engineers, March 2019

# Fault Currents, Circuit Breakers, and a New Method for X/R Calculations in Parallel Circuits

Amir Norouzi, *Schweitzer Engineering Laboratories, Inc.*

**Abstract**—This paper first provides an analysis of the asymmetrical characteristics of fault currents and reviews the fundamental concepts in the ANSI/IEEE symmetrical rating method for circuit breakers. Next, the paper examines the problem of determining a system's X/R when there are parallel circuits at the fault, where there is no single X/R precisely describing the current. The paper develops a rigorous method to construct and algebraically solve the differential equation of the fault current, which provides its accurate transient component. Illustrative examples are provided, and the concept of a variable X/R is proposed to consider the actual impact of the transient current.

## I. INTRODUCTION

Unlike load flow currents, fault currents involve transient components that cannot be ignored when dealing with circuit breaker ratings. Although, compared to the steady state current, the transient part of the fault current is short-lived and lasts only a few cycles, it needs to be included in the overall current as circuit breakers operate within a similar time frame when the transient part is still very much alive.

This transient current is generally modeled as an exponentially decaying dc current with a time-constant ( $\tau$ ) determined by the X/R ratio of the power system at the fault point, namely,  $\tau = \frac{X/R}{\omega}$ . Hence, X/R is a critical number that specifies the amount of transient current, in addition to the steady state ac current, when a circuit breaker attempts to open its contacts. In industry standards such as ANSI/IEEE the transient dc current is expressed as an equivalent additional *rms* value that a circuit breaker must interrupt; thus, the breakers are rated purely based on ac symmetrical currents.

The idea of the transient current being considered as a single decaying dc term with a single time-constant originates from first order circuits with one R and one L component, as described in the following sections. The transient current in

such circuits has a general form of  $I_0 e^{\frac{-\omega t}{X/R}}$  where  $I_0$  is the maximum dc current and  $X = L\omega$ . In higher order systems though, such as when there are parallel branches contributing to the fault current, the transient current consists of multiple components, each with a different time-constant which cannot mathematically be combined as a single exponential decaying term. To handle this problem, two assumptions have traditionally been made, even if not stated explicitly. One is that, regardless of the complexity of the circuit, the transient current can be expressed with only one time-constant. What naturally follows is that each parallel branch would provide a transient current with the same time-constant, otherwise they

cannot be all combined to one exponential term. Hence, circuit reduction techniques such as Thévenin method may be used which would provide system's X/R. The other assumption is that the transient current is essentially a decaying dc current, in the sense that it is a decaying constant, which strikes as obvious in an RL circuit. We will, however, see that while these two assumptions are accurate in first order circuits, they don't necessarily hold true in higher order circuits where there are multiple parallel branches.

The problem in higher order circuits is twofold: First, finding the exact components of the transient current will require setting up and solving a differential equation, which is a daunting task if tackled directly. We cherish our phasor analysis tools because they help us avoid dealing with differential equations by reducing the steady state sinusoidal calculations to algebraic operations. But phasor analysis won't be helpful here as transient response is automatically ignored. Second, even if the exact components of the transient current are calculated, we would still need to come up with a single exponential term to be able to use the concept of symmetrical-based rating for circuit breakers, which is based on one X/R value.

The paper first reviews the basic concepts in fault current characteristics and formulation with application in circuit breaker rating. Basic ideas used in ANSI/IEEE breaker rating structure and calculations are also surveyed. To calculate the complete transient component of the fault current, a general circuit configuration is set up to describe a system with multiple parallel branches contributing to the current at the fault point. By deriving the differential equation of the fault current for some lower order circuits, the pattern of the equation for an  $n$ th order system, with  $n$  parallel branches, is identified. This is the first major step in finding the accurate transient components of the fault current in a circuit with parallel branches.

Next, some basic concepts in linear ordinary differential equations (ODE) are briefly reviewed and are then applied to the ODE of our system. It is shown how the mathematical concepts of ODEs, such as characteristic equation and initial conditions, are applied to an ac circuit with sinusoidal inputs to calculate the complete transient fault current. The key is that in power systems the coefficients of the fault current ODE are real and constant numbers, and this is what makes it possible to reduce the procedure of solving the ODE to purely repetitive algebraic operations. Since these algebraic operations can be computerized, the order of the circuit and complexity of the calculations will be of little concern, the same way that, regardless of the size of a power system, load flow calculations

can be reduced to some matrix operations for computerized processing.

As it turns out, in higher order systems, that is in circuits with higher number of parallel branches at the fault, it is possible that the transient component is no longer a pure decaying dc current, but either a decaying sinusoidal current or a combination of decaying dc and sinusoidal terms. This will have major implications for our fundamental assumption that the transient current can be expressed with a single X/R and that this X/R can be used to size an appropriate circuit breaker. The actual fault current calculated as such may now be substantially larger than what is estimated by the X/R method at the time of circuit breaker opening, potentially surpassing its interrupting capability.

To properly account for the actual transient current due to parallel branches and still be able to use the concept of, and ratings associated with, a single X/R, we may define a variable X/R where it is no longer a constant value over the duration of fault, but would vary to properly simulate the actual transient current at any time. Of interest, is the breaker's contact parting time when we need to know the magnitude of the current that must be interrupted. To do this, we can calculate the transient current at any time and then work backwards to come up with an X/R that would provide the same transient current at that moment had the fault started with that X/R. The paper provides the basis and calculations for this variable X/R. The variable X/R can also be used in other applications such as current transformer sizing where saturation considerations have important consequences in protective relaying design.

## II. CHARACTERISTICS OF FAULT CURRENTS

Power systems are typically inductive and can be modeled with RL circuits, so an analysis of transient response of a simple RL circuit can provide some insight into the characteristics of fault currents.

Fig. 1 shows an RL circuit with a sinusoidal voltage source,  $V_m \sin(\omega t + \theta)$ . A fault occurs when the switch is closed. Since the relationship between an inductor's voltage and current involves a derivative ( $v = L \frac{di}{dt}$ ), we will have to deal with some form of a differential equation in calculating the current. Here, the equation describing the fault current is a simple KVL equation:  $L \frac{di(t)}{dt} + Ri(t) = V_m \sin(\omega t + \theta)$ . This is a first order, linear ordinary differential equation as it involves only the first derivative and no partial derivatives. Solution techniques of ODEs will be briefly discussed later, but it can be shown that the complete solution to the above equation is per below:

$$i(t) = \underbrace{\frac{V_m}{Z} \sin(\omega t + \theta - \alpha)}_{\text{steady state}} - \underbrace{\left[ \frac{V_m}{Z} \sin(\theta - \alpha) - i(0) \right]}_{\text{transient}} e^{-\frac{\omega t}{X/R}} \quad (1)$$

In the above equation,  $Z = \sqrt{R^2 + (L\omega)^2}$ ,  $\alpha = \text{tg}^{-1} \frac{L\omega}{R}$ , and  $\omega = 2\pi f$ . It is assumed that the initial condition of the system,

that is the current right at the time of closing the switch, is  $i(0)$ . Complete solution of ODEs requires knowing the initial conditions of the dynamic system, which will be additionally discussed later. Although initial conditions determine the specific response of a dynamic system, they don't change the general formulation or characteristics of the system's response. In our circuit, assuming an initial current of zero will only change the amplitude of the transient part of the current while the general formulation or nature of the response remains the same.

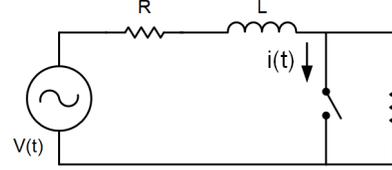


Fig. 1. Simple RL Circuit

Inspecting (1) shows that the current has two parts. One is transient in nature as it will diminish over time due to the exponential decaying term. The other part is a sinusoidal term that will last if the input voltage remains in the circuit; this is the steady state response of the system. The steady state response is basically the part that we obtain from our well-known phasor analysis where its amplitude,  $\frac{V_m}{Z}$ , is calculated from the system's ac impedance, and the angle  $\alpha$  is the phase shift determined by the impedance angle ( $\alpha = \text{tg}^{-1} \frac{X}{R}$ ).

The transient part of the current has a time-constant of  $\tau = \frac{X/R}{\omega}$  which is solely determined by the system's X/R for the specific frequency of the power system. The larger the X/R, the longer it takes for the transient current to fade away. Intuitively, from our ac analysis knowledge we know that a more inductive circuit will take more time for its current to diminish once the source of the current is removed; and a more inductive circuit means a larger X/R. This is how X/R plays an important role in the characteristic of the transient component of the fault current. It is also important to notice that the time-constant of the system is solely determined by the circuit's elements R and L; no other variable such as voltage amplitude and phase or the initial conditions of the circuit has any impact on the time-constant. In other words, the time-constant is a physical characteristic of the system.

While the rate of decay of the transient current depends only on the circuit elements, its amplitude depends on input voltage, circuit elements, and, importantly, on the initial conditions of the system. Even for a specified set of circuit elements and input voltage, the amplitude of the transient current can substantially vary based on the initial conditions of the system, which is the initial current in the inductor at the time of fault. For example, it is possible to have no transient current if the switch is closed at a time such that the transient current will have zero amplitude.

For fault studies and circuit breaker rating, what matters is the highest possible current that each piece of equipment must withstand and, in the case of a breaker, interrupt. The transient current in (1) becomes maximum when the initial current is zero and the angle  $(\theta - \alpha = 90^\circ)$  is any integer multiple of  $90^\circ$ , which would make the sine function  $\pm 1$  for a positive or negative maximum transient current. Hence, for a negative maximum transient current  $(\theta - \alpha = 90^\circ)$  the highest fault current becomes:

$$i_{\max}(t) = \frac{V_m}{Z} \left[ \cos(\omega t) - e^{\frac{-\omega t}{X/R}} \right] \quad (2)$$

Fig. 2, created in MATLAB, shows  $i_{\max}(t)$  in per unit of  $\frac{V_m}{Z}$  for  $\frac{X}{R} = 17$ , equivalent to a time-constant of 45ms, which is a standard X/R in ANSI/IEEE standards for circuit breaker rating. The time axis is in power system cycles to provide a more insightful view into the transient current. The initial condition of the system is selected in such a way that the transient component has the maximum amplitude of one per unit at the beginning of the fault, hence the circuit breaker will see the highest possible current that may occur. Generally, it is desired for a fault to be cleared within 5 cycles after occurrence, and hence we can see why we need to pay special attention to the transient current. Besides, X/R is often greater than the standard value of 17, which makes the transient current even stronger when a breaker attempts to clear the fault.

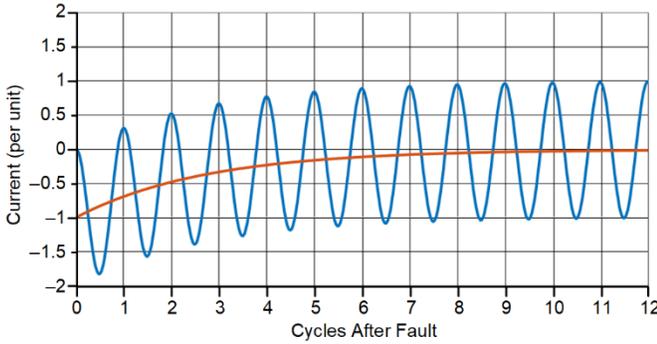


Fig. 2. Fault Current Characteristics

In (2), the transient component is a decaying constant current and therefore it is typically called the dc component of the fault current. We should be aware that, for higher order systems, the transient component is not necessarily a decaying constant. For these systems, it is possible that the transient current could be a decaying sinusoidal term, and hence the term “transient” is a more general description than “dc.”

Phasor analysis, although a very powerful tool, is completely blind to the transient response of the system and to the characteristics that have been discussed so far. In fact, ignoring the transient part of the response is what makes the phasor analysis possible. The power and magic of phasor analysis is that it reduces the sinusoidal ac analysis to purely algebraic operations, which is enormously easier than dealing

with differential equations. Besides, these repetitive operations can be easily implemented in digital computers which makes it possible to perform load flow and other steady state analyses for any size and degree of complexity in power systems. As we will see, in the standard circuit breaker sizing computations the transient current is estimated as a percentage of the steady state fault current, making it possible to use the phasor analysis as the basis for the total fault current calculations.

In the next section, we discuss how the system’s X/R can be used to establish a simple basis for calculating the transient current of the fault and use that to determine interrupting capabilities of circuit breakers.

### III. FAULT CURRENTS AND CIRCUIT BREAKER RATING

#### A. Basic Concepts

Perhaps the most important rating of a circuit breaker is its interrupting capability. Simply put, it is the highest current, expressed in ac symmetrical *rms* value, that a circuit breaker can safely interrupt when it attempts to do so. Hence, the effect of the transient component of the fault current is expressed in *rms* value, so that the breaker’s rating can be put into a single ac *rms* value.

While we need to include the effect of the transient current in the interrupting capability of circuit breakers, we also want to avoid dealing with the system’s differential equation. The first thing to agree on is that only the highest possible amplitude of the transient current needs to be taken into consideration; once the breaker is rated for the highest current it will also do well under all other conditions. We saw how this led to (2), which provides the maximum fault current. Second, we would like to rate circuit breakers based on only one *rms* current that already includes the effect of the transient component, and hence not have to deal with two types of currents.

We can simply calculate the *rms* current obtained from (2) and express it as one current value that includes the effects of both ac and transient components. From basic signal analysis, we know that the *rms* value of a signal consisting of a sinusoidal

and a dc component is  $\sqrt{a_o^2 + \left(\frac{a_1}{\sqrt{2}}\right)^2}$ , where  $a_o$  is amplitude of

the dc component and  $a_1$  is the amplitude of the sinusoidal component. In (2), the amplitude of the ac component is

$a_1 = \frac{V_m}{Z}$  and the amplitude of the dc component is

$a_o = \frac{V_m}{Z} e^{\frac{-\omega t}{X/R}}$ . By designating the *rms* value of the steady state

current as  $I_{\text{sym}} = \frac{V_m}{\sqrt{2}Z}$ , the *rms* value of the total current,  $I_{\text{asym}}$ ,

can be expressed as  $\sqrt{I_{\text{sym}}^2 + \left(\sqrt{2}I_{\text{sym}} e^{\frac{-\omega t}{X/R}}\right)^2}$  or:

$$I_{\text{asym}} = I_{\text{sym}} \sqrt{1 + 2e^{\frac{-2\omega t}{X/R}}} \quad (3)$$

Equation (3) provides the total *rms* current of (2) due to both ac and dc components, where  $I_{\text{sym}}$  is the steady state *rms* current and  $I_{\text{asym}}$  is total *rms* current. The total *rms* current is also called *asymmetrical* current, because it involves both ac and transient components of the current. For practical purposes, it is also customary to express (3) as  $I_{\text{asym}} = I_{\text{sym}} \sqrt{1 + 2 \left( \frac{\text{dc}\%}{100} \right)^2}$  where

dc% is calculated as  $\text{dc}\% = 100e^{\frac{-\omega t}{X/R}}$ .

Thus, for the first order circuit of Fig. 1 the total maximum fault current is obtained as a function of time by knowing only the steady state current, from phasor analysis, and the system's X/R. This is perfect, because instead of a differential equation we now need to deal with only a simple exponential formula to calculate the total current that a circuit breaker must interrupt at any certain time.

### B. Interrupting Capability of a Circuit Breaker

The total asymmetrical current from (3) is the current that a circuit breaker must interrupt at its contact parting time. Per IEEE C37.04, contact parting time is the sum of ½ cycle, as the minimum relay operation time, and the minimum operating time of the breaker. For example, contact parting time, including ½ cycle for relay operation, is assumed as 1.5 cycles for 2-cycle breakers, 2 cycles for 3-cycle breakers, and 3 cycles for 5-cycle breakers. If the contact parting time is different from the above-mentioned assumed times, for example due to faster or slower relay operation, the required asymmetrical interrupting capability should be accordingly adjusted per (3).

In ANSI/IEEE standards of C37.010 and C37.04, the circuit breaker rating calculations are based on a time-constant of 45ms, corresponding to  $\frac{X}{R} = 17$  at 60 Hz. The breaker rated interrupting capability is expressed in symmetrical *rms* current while the impact of the transient component of the current is already included in the rating for any  $\frac{X}{R} \leq 17$ . For example, for

a contact parting time of 3 cycles, a circuit breaker is required to be able to interrupt a fault current with  $e^{\frac{-0.05\omega}{17}} = 32.9\%$  of dc component, and hence the total asymmetrical current that the circuit breaker must be capable of interruption is  $\sqrt{1 + 2(0.329)^2} = 1.10$  times of its specified symmetrical short circuit rating. However, if the circuit breaker is to be applied in

a system with  $\frac{X}{R} = 40$ , the dc component will be  $e^{\frac{-0.05\omega}{40}} = 62.4\%$  of the steady state current and thus the total asymmetrical current that the breaker must be capable of interruption is  $\sqrt{1 + 2(0.624)^2} = 1.33$  times its symmetrical short circuit rating. Therefore, the steady state fault current must be first multiplied by  $\frac{1.33}{1.10} = 1.21$  and then compared to the symmetrical short circuit rating of the breaker to ensure there is enough capability.

This is the basis of symmetrical rating of circuit breakers in ANSI/IEEE standards [1] [2]. If  $\frac{X}{R} \leq 17$  the effect of the dc (and ac) decrements are already included in the rating calculations of the circuit breaker and it can interrupt the additional dc current as well, although it is not explicitly mentioned in the breaker's short circuit rating. In such cases, 100% of the breaker's capability can be used at the standard minimum contact parting time. Also, per the ANSI/IEEE standards, if the calculated steady state fault current,  $I_{\text{sym}}$ , is below 80% of the breaker's short circuit rating, there is no need to calculate the system's X/R and the breaker can be applied to any system. This method is called "E/X Simplified Method" in C37.010-1964 and after, as the steady state current is calculated from dividing the voltage, E, by the reactance, X.

However, if the system's X/R is greater than 17, or if the calculated E/X steady state current is greater than 80% of the breaker's short circuit rating, the effect of the dc component needs to be computed more accurately. For simplification of the calculations, the basic idea in C37.010 is that multipliers, provided in standard curves, need to be applied to the calculated E/X current to consider the fact that the dc component of the fault current will be larger, at the time of contact parting, for greater X/R. These curves were produced by considering the dc component calculation in (3) as well as the impact of ac decrement (due to the dynamics of  $X_d$  of synchronous generators). These multipliers are typically between 1 and 1.25 which are applied to the calculated steady state fault current for a specific contact parting time and X/R greater than 17.

In short, if the system's X/R is below 17 all that needs to be done is to make sure that the calculated E/X current is not larger than 80% of the breaker's interrupting capability. If  $X/R > 17$  or the steady state fault current is larger than 80% of the breaker's interrupting capability, then the ANSI/IEEE curves of C37.010 should be used to find a multiplier and apply it to the E/X current to ensure that the short circuit rating will still be greater than the adjusted steady state fault current.

As another example, let's consider the impact of the relaying operation time faster than ½ cycle, as this is the minimum operating time considered in the standard contact parting times. For a contact parting time of 3 cycles (5-cycle breaker) which also includes ½ cycle relaying time, we want to reduce the relaying operation time to ¼ cycle and calculate the impact on the total asymmetrical current at the new contact parting time.

We can use (3) with  $\frac{X}{R} = 17$ , and two different contact parting times of 3 (old) and 2.75 (new) cycles. The new asymmetrical current compared to the old current is:

$$\frac{I_{\text{asym\_new}}}{I_{\text{asym\_old}}} = \sqrt{\frac{1 + 2e^{\frac{-2\omega \cdot 2.75}{17 \cdot 60}}}{1 + 2e^{\frac{-2\omega \cdot 3}{17 \cdot 60}}}} = 1.018$$

This means that with relay minimum operating time dropping to half of its assumed value, that is from  $\frac{1}{2}$  to  $\frac{1}{4}$  of a cycle, the total asymmetrical current will increase by 1.8% at the new contact parting time with  $\frac{X}{R}=17$ . The same calculation for a system with  $\frac{X}{R}=40$  yields an increase of 1.77% in the total asymmetrical current due to shorter relaying time. Hence, the increase in the total current due to faster relaying time is marginal, as the reduced operation time of about 4ms is small compared to the time-constant of power system with typical values of 30–200ms.

That the minimum relaying time could be below 0.5 cycle has been acknowledged in the latest C37.010-2016 standard, in which IEEE recommendation is that if a circuit breaker is applied below 80% of its symmetrical rating, faster relaying time below 0.5 cycle has no significant impact on the breaker's interrupting capability. For application above 80% of breaker's symmetrical rating, consultation with breaker's manufacturer is suggested by IEEE.

**Historical Note:** C37.5-1953 provided a simple method to calculate the total fault current for comparison with a breaker's interrupting capability, a rating that was based on *total* current, as opposed to symmetrical current. A breaker's total interrupting rating was composed of the combined ac and dc components and the calculated total fault current was compared to the total interrupting capability of the breaker. The system's X/R and breaker's actual contact parting time were not considered in the rating calculations. In early 1950's and with power systems becoming more complex, AIEE (IEEE's predecessor) started developing a new rating structure based on symmetrical current [3] [4], culminating in the publication of C37.04-1964 and C37.010-1964. The goals of the new rating structures were to simplify the rating, bring the American standards closer to their IEC counterparts, and include X/R and contact parting time in the rating calculations [5]. So, while in the old total-based rating the total asymmetrical fault current was compared to a breaker's total short circuit rating, in the new symmetrical-based rating the symmetrical E/X current is adjusted with multipliers and compared to a breaker's symmetrical rating, knowing that the asymmetrical part of the current is already included in both the fault current and the breaker's rating. A factor,  $S$ , was also defined as  $S = \frac{I_{\text{asym}}}{I_{\text{sym}}}$  that

would determine the relationship between symmetrical rating of a circuit breaker and its required asymmetrical capability, although in practice it became redundant; that relationship was included and considered in the multiplying factors provided in the standard, and therefore the  $S$  factor was removed from C37.04 and C37.010 standards in 1999. Today all ratings and calculations related to circuit breakers are based on symmetrical currents.

#### IV. FAULT CURRENTS IN HIGHER ORDER CIRCUITS

Looking at the concepts and calculations that eventually led to (3)—the basis for the asymmetrical fault current—it is worth

recalling that everything started out with a simple RL circuit. We then calculated the complete fault current and analyzed it to come up with generalized concepts that can help us determine the maximum expected asymmetrical fault current. We were then able to obtain the total fault current as a function of only time and the system's X/R. This is a simple and very effective method as the transient response was assumed to contain only one component: a decaying dc current with a single time-constant. The power system isn't always a simple RL circuit; there are often multiple sources contributing to the fault current. The underlying assumption in using the above method for higher order inductive systems is that any system can be reduced to an equivalent first order RL circuit, at which time our method can be applied.

Although reduction techniques, such as Thévenin equivalent, can be applied to obtain steady state ac response of a system with no accuracy concern, it is not equally applicable to transient response. A higher order circuit, which is a circuit with higher order differential equation, will produce a transient response with multiple components, each with a different decaying time-constant. For example, the fault current in a circuit with three parallel branches contributing to the fault, may include a transient current in the general form of  $i_{\text{dc}} = I_1 e^{k_1 t} + I_2 e^{k_2 t} + I_3 e^{k_3 t}$ . This is a combination of three decaying currents. Mathematically, this current cannot be reduced to one decaying current having a single time-constant. In other words, no  $I$  and  $k$  exist such that  $i_{\text{dc}}$  can be expressed as  $I e^{kt}$  for all  $t$ , except when  $k_1$ ,  $k_2$ , and  $k_3$  are all the same. So, the Thévenin equivalent of a system, having one R and one L and therefore one time-constant, cannot possibly provide the same transient current as the original circuit with three transient components. Intuitively speaking, each component of  $i_{\text{dc}}$  decays at a different rate and no single decaying rate can exactly reproduce the same effect.

While in second and third order systems the transient current is typically still a combination of multiple decaying dc currents, it may be more complex in higher order circuits, and we would need to find the actual transient current by solving the associated differential equation. These higher order ODEs will become extremely complex and any attempt to directly solve them will require substantial computing resources, including hardware and software. In this section a general procedure is developed that can be used to solve the high order ODEs, in a computationally efficient way, by reducing the solution to algebraic operations. This is possible because the power system is a linear and time-invariant dynamic system whose characteristics and components don't change with time, hence leading to ODEs with constant coefficients. Besides, the inputs to power system are sinusoidal functions and we already have efficient tools to deal with them.

Next, we will briefly review some major concepts in Ordinary Differential Equations to assemble the techniques that are required to obtain the general solution of the fault current ODE in higher order systems.

A summary of the steps and procedures described in the following Subsections A, B, C and D are provided in Subsection E for quick reference and review.

### A. A Brief Review of ODEs

A linear ODE is an equation containing a function,  $y$ , and its derivatives in a general form of:

$$a_n y^n + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = f(t) \quad (4)$$

In general, the coefficients  $a_i$  can be time-dependent.  $y^{(k)}$  is the  $k$ th derivative of  $y$ , which is a function of time only. The equation is called linear because the function  $y$  or its derivatives appear only once in each term. This is an  $n$ th order ODE, as  $n$  is the highest derivative in the equation. It is also ordinary because there are no partial derivatives involved.  $f(t)$  is the forcing function or the input to the system. ODEs typically describe a dynamic system such as a mechanical or electrical system.

Equation (4) is called *homogeneous* when there is no input to the system; that is, when  $f(t) = 0$ . A homogeneous equation describes a dynamic system's response to its initial conditions only and in the absence of any forcing function, such as the response of an RL circuit with some initial current and no input voltage. Homogeneous response is a characteristic of the system and depends only on the system's structure and its initial conditions, hence it is also called the natural response. In our RL circuit, the homogeneous response depends only on the values of  $R$ ,  $L$ , and the initial current.

A special class of ODEs involve those with constant  $a_i$  coefficients. In power systems, we generally deal with ODEs with constant coefficients, as the systems elements such as  $R$  and  $L$  are time-invariant. This class of ODEs is much easier to solve than when the coefficients are time-varying. In constant-coefficient ODEs the polynomial equation:

$$a_n m^n + a_{n-1} m^{n-1} + \dots + a_1 m + a_0 = 0 \quad (5)$$

is called the *characteristic equation*, which has the same order and coefficients as the original ODE. So, for an  $n$ th order ODE the characteristic equation will be an  $n$ th order polynomial with constant coefficients. As the name suggests, the characteristic equation is important because its roots,  $m$ , determine the general solution to the homogeneous equation. In other words, it provides the general response to the initial conditions of the system.

The characteristic equation has  $n$  roots, which could be real, complex, or a combination of real and complex numbers; some of the roots could be repeated too, which is not discussed here as this is not expected to occur in power systems. For any real root  $\alpha$ , the term  $Ce^{\alpha t}$  is a component of the response of the homogeneous equation. Similarly, if  $\lambda \pm j\mu$  is a complex root, then  $Ae^{\lambda t} \cos(\mu t) + Be^{\lambda t} \sin(\mu t)$  is a component of the homogeneous response. The reason that the complex root is shown as a complex conjugate pair is that in polynomial equations it is shown that if all the coefficients in (5) are real numbers, which is the case in electric circuits, then any complex conjugate of a complex root is also a root of the equation; therefore, all complex roots will be in conjugate pairs, each pair counted as two roots. The constants  $C$ ,  $A$ , and  $B$  in the above response terms are determined by the initial conditions of the system. For example, if a fourth order characteristic equation of a homogeneous ODE has two real roots,  $\alpha_1$  and  $\alpha_2$ , and one pair

of complex conjugate roots,  $\lambda \pm j\mu$ , then the general solution is:  $y_h(t) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t} + Ae^{\lambda t} \cos(\mu t) + Be^{\lambda t} \sin(\mu t)$ , where  $y_h(t)$  is the homogeneous response. By knowing the initial conditions of the system and its derivatives at  $t = 0$ , the unknown coefficients can be calculated.

When the forcing function  $f(t)$  is nonzero, the ODE is called *nonhomogeneous*, which describes a dynamic system with inputs, such as when there is a voltage source in an RL circuit. There is a fundamental relationship between the solutions to the nonhomogeneous equation and the solutions to the corresponding homogeneous equation [6]. If  $y_h(t)$  is the general solution to the homogeneous ODE and  $y_p(t)$  is any specific function that satisfies the nonhomogeneous equation, then  $y(t) = y_h(t) + y_p(t)$  is a general solution to the nonhomogeneous equation. Besides, every solution is expressed in this sum. In other words, the general response of a dynamic system with forced input that can be described by (4) is the sum of its general natural response and a specific response of the system.

Although finding a specific solution to the nonhomogeneous ODE is not a trivial task, there are relatively simple methods to find a specific solution for ODEs with constant coefficients and when the forcing function is of certain types such as sinusoidal inputs. However, in ac circuit analysis we have a great tool that readily provides us with the specific solution: phasor analysis. Hence, we are left with finding the natural response of our ac system, but before that we first need to be able to set up the ODE of the ac circuit. Next, we are going to find out how the system's ODE can be established and how we can apply the basic properties that were discussed here to find the complete response of an ac system with parallel circuits.

### B. Application in AC Circuit Analysis

In our search for the transient component of the fault current in second and higher order systems we need to find the complete solution to the ODE that describes the fault current. The solution of this ODE is obtained by adding the general response to the associated homogeneous equation, and a specific solution that satisfies the nonhomogeneous equation. We then need to find the unknown coefficients of the total response by applying the initial conditions of the system to the solution.

One way to think of the specific solution to the nonhomogeneous equation is that it is the response when the homogeneous solution,  $y_h(t)$ , is zero in the total solution of the system,  $y(t) = y_h(t) + y_p(t)$ , which is when all the coefficients in  $y_h(t)$  are zero. This situation happens when the initial condition of the system is equal to the steady state conditions at  $t = 0$ . Under these circumstances, no transient will be experienced by the system and the steady state conditions will be immediately established upon application of the inputs. Therefore, the specific solution,  $y_p(t)$ , is in fact the steady state response. Another way of reasoning for this, is that when  $y_h(t)$  diminishes, after a while only  $y_p(t)$  is left in the total response; since the system's response is the steady state solution after the transient response is diminished, we conclude that  $y_p(t)$  is in fact the steady state solution. This is true for a system whose ODE has constant or time-independent coefficients which makes it a

time-invariant system; and in a time-invariant system, as opposed to time-dependent systems, there is only one specific response which is the steady state solution.

In short, the specific response of the nonhomogeneous ODE of an ac system is in fact the steady state solution which can be obtained from phasor analysis, a powerful tool that can handle ac systems with any complexity. We may describe the steady state fault current as  $I_m \cos(\omega t + \theta)$ , where  $I_m$  is the amplitude of the steady state fault current and  $\theta$  is its phase angle. In phasor notation, this current is represented as  $I_m \angle \theta$ .

Now we need to obtain the general solution to the homogeneous ODE of the circuit to add it to the steady state response. But before that we first need to set up the differential equation that describes the fault current.

Fig. 3 shows a general circuit configuration from a fault viewpoint. It includes  $n$  parallel contributing branches, a fault resistance,  $R_f$ , and a fault inductance,  $L_f$ . The fault point could be, as an example, considered on a transmission line. Each parallel branch has a source,  $V_k$ , a resistance,  $R_k$ , and an inductance,  $L_k$ , which could be the result of circuit reduction techniques on that part of the power system. A similar circuit configuration could be on the right side of the fault point too, contributing to the fault current,  $i_f$ , where similar calculation methods will be applicable. This circuit is used as the general representation of the power system at a fault where multiple parallel branches, each assumed to be inductive, add to the fault current.

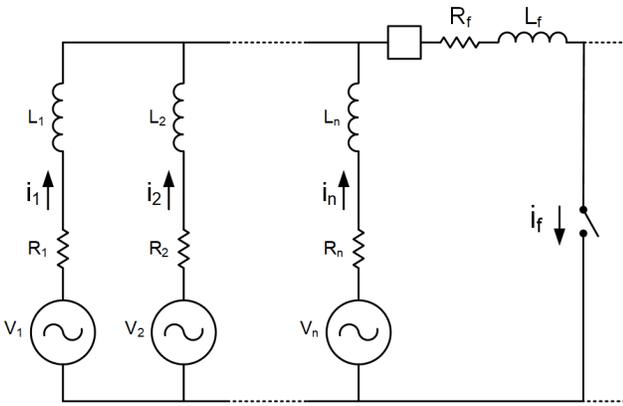


Fig. 3. General Configuration of a Faulted System With Parallel Branches

We are looking only for the homogeneous response for the fault current since the steady state solution will be obtained from phasor analysis. Recall that the homogeneous equation has no forcing function or input, and the system responds to its initial conditions only. Hence, in deriving the ODE, all voltage sources are ignored. Deriving the homogeneous ODE for fault current in Fig. 3 will quickly become extremely complicated for three and higher parallel branches. The idea here is to obtain the ODE, starting from the lowest number of branches, and try to find a repetitive pattern to set up the ODE for any number of branches.

Appendix I provides the procedure to find the homogeneous ODE of the fault current in Fig. 3 and a general method to construct the ODE for any number of parallel branches. The equation is an  $n$ th order ODE, equal to the number of parallel

branches. It has constant coefficients, each being a function of the circuit  $R$ 's and  $L$ 's only. Once we know the number of branches and the elements in each of them, as well as the fault's resistance and inductance, the homogeneous ODE can be set up by some repetitive products and sums of circuit resistances and inductances. An example of a third order circuit is provided in Appendix I. This is a first major step, which makes it possible to construct the circuit's ODE without any need to get involved with conventional KCL or KVL equations. The procedure is purely algebraic and can be computerized with simple operations.

Once the homogeneous ODE is set up, the first step for calculating its general solution is to solve the associated polynomial characteristic equation, the roots of which determine the general solution of the fault current's ODE. The characteristic equation is an  $n$ th order equation with  $n$  roots, which may be a combination of real and complex numbers. General software tools can be used to solve this polynomial equation.

### C. Characteristics of the Homogeneous Response

As can be seen in Appendix I, the coefficients of the fault current's ODE are all real positive numbers. Per Descartes's Rule in algebra, the number of positive roots in a polynomial with real coefficients is equal to the number of sign changes (positive to negative or vice versa) of the polynomial coefficients. Since the coefficients of our ODE are all positive with zero change in their signs, there will not be any positive real roots; the roots are either real negative numbers or complex conjugate pairs. This is what we expect in circuit analysis: positive real roots will create an exponential term with a positive exponent, which means the response will only increase with time, making the system unstable. While this rule guarantees decaying exponential terms associated with the real roots, it doesn't guarantee that the complex roots will have negative real parts that will appear in the exponent of the corresponding response, as in  $Ae^{\lambda t} \cos(\mu t) + Be^{\lambda t} \sin(\mu t)$  for complex roots of  $\lambda \pm j\mu$ . But we intuitively expect the complex roots to have negative real parts, making a decaying homogeneous response; we cannot think of a set of initial conditions in an electric circuit that would make the natural response an ever-increasing unstable current.

Since the complex roots will only appear in complex conjugate pairs, for any circuit of an odd order there must exist some real roots, while for even order circuits there is a possibility that all roots could be complex conjugate pairs. Nonetheless, for a general case of combined real and complex roots for the characteristic equation, the transient response of the circuit will have a general structure of  $\sum I_k e^{\alpha_k t} + \sum [A_i e^{\lambda_i t} \cos(\mu_i t) + B_i e^{\lambda_i t} \sin(\mu_i t)]$ . Here,  $\alpha_k$  are all the real roots of the characteristic equation, and  $\lambda_i \pm j\mu_i$  are all the roots in complex conjugate pairs.

The above homogeneous solution, or in fact the transient part of the complete fault current, is only the general solution with unknown coefficients  $I_k$ ,  $A_i$ , and  $B_i$  that need to be determined. These coefficients are determined by the initial

conditions of the electric circuit right at the instant of the fault. The general procedure is to first set up the total response by adding together the transient (homogeneous) and steady state (specific) responses; this is  $y_h(t) + y_p(t)$ , as we saw earlier. Then by applying the initial conditions, and their  $(n-1)$  derivatives, to the total response we can calculate the coefficients of the transient response. This is what we are going to investigate next.

#### D. Initial Conditions and the Coefficients of the Transient Component of the Fault Current

The last step in calculating the complete solution to the fault current's ODE is to find the coefficients of the transient response, which are determined by the initial conditions of the circuit. It is assumed that there is no load current in the faulted transmission line prior to the fault. This provides the first initial condition, which is  $i_f(0) = 0$ .

The number of unknown coefficients in the transient response is equal to the order of the ODE, which is the same as the number of parallel branches in Fig. 3. So, for an  $n$ th order circuit we need  $n$  equations to solve for the coefficients. We already have one in  $i_f(0) = 0$ , as zero load current is considered on the fault path on the transmission line prior to fault; but we need  $(n-1)$  more equations, corresponding to  $(n-1)$  more conditions. The rest of the equations should come from the derivatives of the initial conditions of the circuit as this is all we can know; we don't know the exact dynamics of the system beyond  $t = 0$ . Hence, we need to find the values for all  $(n-1)$  derivatives of  $i_f(0)$ , that is  $i_f'(0), i_f''(0), \dots, i_f^{(n-1)}(0)$ . It is important to note that even when the initial currents in the circuit, including  $i_f(0)$ , are zero, it doesn't mean that their derivatives are zero too [7]. The circuit currents are changing rapidly and have values for their derivatives, even if they are zero at the instant of the fault. Think of a sine function which is zero at a specific time, but its derivative, a cosine function, is not zero at that moment. So, all the derivatives need to be obtained from the initial conditions of the circuit, right at the time of fault.

A curious question may now be asked: since a fault can occur at any time during the steady state pre-fault conditions, corresponding to many initial conditions of the system at the time of the fault, what time do we need to consider in calculating the transient component of the fault current? Depending on the initial conditions that we pick, a specific transient response is obtained. We know that the total fault current is expressed as:

$$i_f(t) = i(t)_{\text{transient}} + \overbrace{I_m \cos(\omega t + \theta)}^{\text{steady-state}} \quad (6)$$

The second term of the above equation is the steady state fault current. We are interested in the maximum of  $i_f(t)$  because what we are trying to determine is the maximum current that a circuit breaker must interrupt, and other system components must withstand. The transient current is a decaying current and so to have the maximum fault current, it should start from the maximum transient current, that is when  $i(0)_{\text{transient}}$  is

at its extremum (highest positive or negative); after that, the transient component will only decrease. When  $i(0)_{\text{transient}}$  in (6) is at its extremum value, the steady state current will have to be at its opposite extremum to satisfy the requirement that the total fault current at  $t = 0$  be zero. For example, if  $i(0)_{\text{transient}}$  is a positive maximum, the steady state current must be at an equal but negative minimum value. Since the fault can occur at any moment, including a time when the steady state component starts at its extremum value, we can conclude that 1) the extremum value of  $i(0)_{\text{transient}}$  can get to the same value of the extremum value of the steady state fault current, and 2) the extremum of the  $i(0)_{\text{transient}}$  cannot be higher than that of the steady fault state current, otherwise the requirement of  $i_f(0) = 0$  will be violated. This means that the extremum value of  $i(0)_{\text{transient}}$  is equal and opposite to  $i(0)_{\text{steady state}}$ , which is in fact  $I_m$ . Going back to the question of what initial conditions we need to consider when calculating the fault current for breaker sizing, we realized that we need to make  $i(0)_{\text{transient}}$  at its extremum value and that this happens when steady state fault current,  $I_m \cos(\omega t + \theta)$ , is at its extremum at the time of fault occurrence. For steady state current to be at an extremum value, the argument of the cosine,  $(\omega t_0 + \theta)$ , should be zero or any integer multiple of  $\pi$ , where  $t_0$  is the time of the fault occurrence. This condition will make the value of the steady state component of the fault current equal to  $\pm I_m$ , and hence

$i(0)_{\text{transient}} = \mp I_m$ . Therefore,  $t_0 = \frac{-\theta}{\omega}$  will make the steady state component at its positive maximum and  $t_0 = \frac{\pi - \theta}{\omega}$  will make it at its negative minimum. Full cycles could be added to  $t_0$  with the same results. Besides, whether we use the time corresponding to the positive or negative extremum will merely change the starting point of the fault current from a positive or negative value, with no impact on the *rms* value of the total current that we are interested in for breaker sizing.

Now that we determined the time that the initial conditions should be evaluated for maximum fault current, we can start calculating  $i_f'(0), i_f''(0), \dots, i_f^{(n-1)}(0)$  and solve for the coefficients of the transient component of the fault current. The key steps are described here, with reference to Fig. 3 as the general configuration of the system. At the fault's point, we have  $i_1(0) + i_2(0) + \dots + i_n(0) = 0$ , where  $i_m(0)$  is the current of the  $m$ th parallel branch at the time of fault, which we consider the reference time,  $t = 0$ , for the fault current calculation. It follows that the same relationship exists for the derivatives of the currents:  $i_1^{(k)}(0) + i_2^{(k)}(0) + \dots + i_n^{(k)}(0) = i_f^{(k)}(0)$  for any  $k$ th derivative of the currents. We have the pre-fault branch currents from phasor calculations, each being  $I_{m_k} \cos(\omega t + \theta_{m_k})$ , equivalent to a phasor representation of  $I_{m_k} \angle \theta_{m_k}$ . All these currents can be evaluated at the time of the fault occurrence, which is  $t_0 = \frac{-\theta}{\omega}$  or  $t_0 = \frac{\pi - \theta}{\omega}$ , where  $\theta$  is the phase angle of the steady state fault current. It should be noted that phasor calculations for pre-fault currents and steady state fault current

have the same time reference, and the time of fault occurrence,  $t_0$ , is a time in the pre-fault reference frame. For the fault current's ODE, the time reference is from the moment that the fault occurs, as the ODE describes the system after the fault and is not valid prior to the fault. So,  $t = 0$  is the time reference for the ODE of the fault current. Since all the branches as well as the fault branch are inductive, and due to current continuity principle in inductors, the current instantly after the fault, shown as  $t = 0^+$ , is equal to the current instantly before the fault, at  $t = 0^-$ .

Unlike the branch currents, their derivatives cannot be directly calculated from their pre-fault equations. The current continuity principle requires that the branch currents remain unchanged at the instant of the fault; however, their derivatives are proportional to the voltage across the inductor of that branch; and there is no continuity requirement for an inductor's voltage. In fact, inductor voltages do change at the instant of the fault as the fault path is added to the system and all the voltages need to follow a new KVL equation.

Appendix II provides a general procedure to calculate the derivatives of the branch currents at the time of fault and then, using the fault current and its  $(n-1)$  derivatives, to calculate the coefficients of the transient current. Some details are involved, and extra attention is required when dealing with the derivatives of the fault current at  $t = 0$  as described in Appendix II. As the calculations involve a system of linear equations, matrix notation can provide a generalized method for computerized implementation of the solution.

This was the last step in finding the complete fault current, comprising a transient and a steady state component. Let's review a summary of the steps and procedures that have taken place so far to solve the ODE of the fault current.

#### E. Summary of the ODE Solution Procedure

We have now a general method to find the complete solution to the fault current's  $n$ th order ODE of Fig. 3. We are interested in the transient part of the current to better understand the actual transient fault current that a circuit breaker must interrupt. Once the homogeneous ODE is set up per Appendix I, with all coefficients being real positive numbers, the solution method includes algebraic operations only. Hence, computer algorithms can be developed to set up the ODE and find the solution. The required data include only circuit components,  $R$ ,  $L$ , and  $V$ . Initial conditions are calculated to provide the maximum fault current for a circuit breaker. The overall procedure can be summarized per below:

1. The circuit model at the fault's point is arranged per Fig. 3, with  $n$  parallel branches contributing to the fault.
2. Steady state fault current is obtained by phasor calculations, represented as  $I_m \cos(\omega t + \theta)$ .
3. The characteristic equation of the fault current's homogeneous ODE is obtained by the method described in Appendix I. The roots of this polynomial equation provide the general formulation of the transient current as

$$\sum I_k e^{\alpha_k t} + \sum [A_i e^{\lambda_i t} \cos(\mu_i t) + B_i e^{\lambda_i t} \sin(\mu_i t)],$$

where  $\alpha_k$  is a real root and  $\lambda_i \pm j\mu_i$  is complex root as a

complex conjugate pair.  $I_k$ ,  $A_i$ , and  $B_i$  are unknown coefficients that need to be determined.

4. The coefficients of the transient current are determined by the initial conditions of the circuit, which are the initial currents and their derivatives at  $t = 0$  as a reference time when the fault occurs. This corresponds to a calculated  $t_0$  in the pre-fault reference time which will make the transient current at its extremum value at the instant of fault occurrence. Appendix II provides the method to find the coefficients of the transient current.
5. The complete fault current is then the sum of the transient and steady state currents, or  $I_{\text{transient}} + I_m \cos(\omega t + \theta)$ .

To see all these steps in action and how they produce a final complete fault current, two illustrative examples are provided in the next section with a third and a fourth order circuit. The transient currents from the accurate calculations are compared to those from the phasor reduction method, and then the impact on the fault current that a circuit breaker must interrupt is evaluated.

## V. ILLUSTRATIVE EXAMPLES

Two examples are provided in this section for accurate short circuit calculations for a third and a fourth order system. In the first example, all roots of the characteristic equation for the fault current are real negative numbers, hence the transient component of the fault current consists of three exponentially decaying constants. In the second example, for a fourth order circuit, there is a pair of complex conjugate roots; this creates a decaying sinusoidal transient current that will be substantially different from the phasor estimation method. In both examples the calculated transient current is compared to the phasor-based dc current. Both examples have the same circuit configuration of Fig. 3.

#### A. Example 1: A Third Order Circuit

Consider the circuit in Fig. 3 with three parallel branches contributing to the fault. Circuit voltages and elements are provided below in *per unit* values. To be more in line with power system units, inductors are provided as reactance values, expressed in multiples of  $R$ ; in fact, the multiplier is the  $X/R$  of each branch. Voltages are expressed in phasor notation.

$$\begin{aligned} R_1 &= 0.0002 & X_1 &= 65R_1 \\ R_2 &= 0.005 & X_2 &= 5R_2 \\ R_3 &= 0.002 & X_3 &= 25R_3 \\ R_f &= 0.0001 & X_f &= R_f \\ V_1 &= 0.98 \angle 0^\circ & V_2 &= 0.97 \angle 15^\circ \\ V_3 &= 1.03 \angle -10^\circ \end{aligned}$$

The parallel circuits represent a combination of strong and weak sources; circuit one can be a large nearby power plant, circuit two an electrically remote small generation source, and circuit three a grid equivalent with normal strength. The fault is mostly resistive with very low resistance, such as a fault on a transmission line very close to a substation.

MATLAB code was developed for the calculation of these examples. Per the steps described in Section IV.E, first the steady state pre-fault currents as well as the steady state fault current are obtained by phasor calculations. Steady state fault current is:  $i_{\text{sym}} = 130.57 \angle -83.22^\circ$ . From the angle of the steady state fault current (in radians), the time for the maximum transient at the instant of the fault ( $-\frac{\theta}{\omega}$ ) is  $t_0 = 3.85\text{ms}$ .

The characteristic equation is obtained from the determinant of matrix  $A$ , per the method described in Appendix I. It has three real negative roots:  $m_1 = -75.52$ ,  $m_2 = -15.15$ , and  $m_3 = -6.08$ . Therefore, the general transient component of the fault current is:  $i_{\text{transient}} = I_1 e^{m_1 t} + I_2 e^{m_2 t} + I_3 e^{m_3 t}$ , with the total current being:  $i_f(t) = i_{\text{transient}} + i_{\text{sym}}$ .

Coefficients of the transient current are determined from the initial currents of the circuit, including  $i_f(0^+) = 0$  and its two derivatives from the procedure described in Appendix II. This will result in  $i_f'(0^+) = 3.604 \cdot 10^3$  and  $i_f''(0^+) = -1.878 \cdot 10^7$ . Coefficients of the transient current can now be calculated, which results in  $I_1 = -37.81$ ,  $I_2 = -20.33$ , and  $I_3 = -72.42$ . Now we have the complete fault current, calculated for the maximum transient component at the start of the fault.

For better visualization, three graphs are provided, produced in MATLAB. Fig. 4 shows the calculated total fault current along with the calculated transient dc and Thévenin-based (phasor-based) dc component of the fault current. All values are in per unit of the amplitude of the steady state fault current ( $I_m$ ). As expected, calculated dc current starts from its extremum value of  $|I_m|$  and the total fault current is zero at the time of fault occurrence. The graph also shows the dc current based on the simplified phasor reduction (Thévenin) method. This current starts from its extremum of  $|I_m|$  per its definition in (3). The gap between the calculated and phasor-based dc currents increases with time until they both drop to zero. Notice that the actual calculated dc current has greater absolute value than the phasor-based estimated dc current, including in third and fourth cycles, when the circuit breaker is expected to interrupt the total fault current. Fig. 5 shows the absolute difference between the two dc components in percentage. We are interested in 3–5 cycles after the fault for typical circuit breaker interrupting time; and we can see that the actual dc current is as much as 25+% greater than the phasor-based estimation. In the next section, we will see how this impacts the *rms* current that a breaker must interrupt.

Finally, Fig. 6 shows the individual dc components of the calculated dc current and how they make the total dc current. All currents are in per unit value of  $I_m$ . It shows how the three currents, each having a different decaying rate, combine to create the total dc current. Such total dc current cannot be accurately modeled with a single exponential decaying current.

As expected, the actual dc component calculated from the individual dc currents will be different from the simplified phasor reduction method. Typically, the roots of the characteristic equation for second and third order circuits are

expected to be negative real numbers; hence the transient component of the fault current, although not exactly following a single time-constant model, will be a smooth decaying current. This example included parallel branches with quite different strengths; that is, very different X/R values. For example, this combination may exist near large generating stations where, as a strong source, it feeds into a nearby substation which is electrically remote from another generating station, making it a weak source for the substation. The difference between the calculated and estimated dc components may be less striking when the sources contributing to the fault are almost similar in strength. In general, the proximity of large generating stations is a prime example of how the transient current could be substantially different from the estimated simplified current. In this example, a 25% increase in the dc component may in fact make the fault current larger than the circuit breaker's interruption rating, as will be discussed in Section VI.

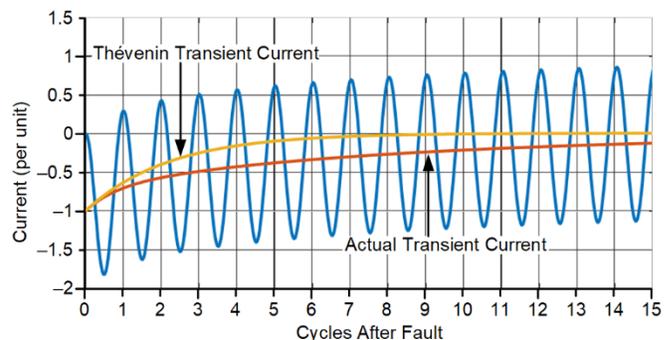


Fig. 4. Total Fault and Transient Currents—Example 1

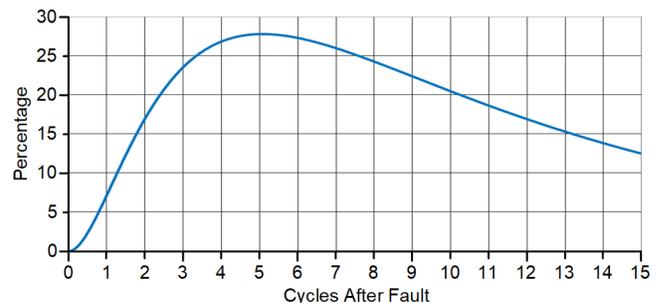


Fig. 5. Difference Between the Accurate and Phasor-Based Transient Currents—Example 1

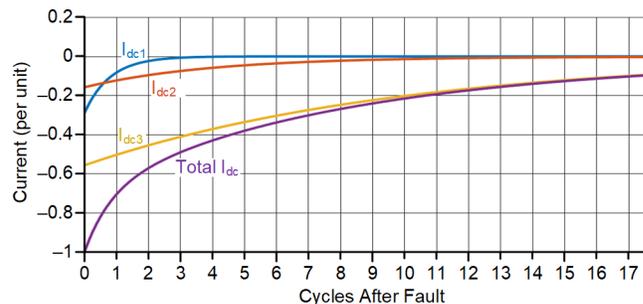


Fig. 6. Components of the Transient Current—Example 1

### B. Example 2: A Fourth Order Circuit

Consider the circuit in Fig. 3 with four parallel circuits with the circuit elements described below in *per unit* values. Each branch reactance is expressed by  $X/R$  of the branch.

$$\begin{aligned} R_1 &= 0.003 & X_1 &= 15R_1 \\ R_2 &= 0.004 & X_2 &= 14R_2 \\ R_3 &= 0.0025 & X_3 &= 30R_3 \\ R_4 &= 0.004 & X_4 &= 15R_4 \\ R_f &= 0.0018 & X_f &= 0.5R_f \\ V_1 &= 0.98\angle 0^\circ & V_2 &= 0.97\angle 15^\circ \\ V_3 &= 1.03\angle -10^\circ & V_4 &= 1.01\angle 25^\circ \end{aligned}$$

The above system has typical  $R$  and  $X$  values of a power system; fault is mostly resistive. A similar MATLAB code was used to calculate the complete fault current with transient component obtained from solving the homogeneous ODE of the fault current. Steady state fault current is:  $i_{\text{sym}} = 62.76\angle -72.20^\circ$ . From the angle of the steady state fault current (in radians), the time of maximum transient at the instant of fault ( $-\frac{\theta}{\omega}$ ) is  $t_0 = 3.34\text{ms}$ .

The characteristic equation is a fourth order polynomial, obtained from the determinant of matrix  $A$ , as in Appendix I. The roots of the characteristic equation are:  $m_1 = -46.25$ ,  $m_{2,3} = -9.88 \pm j40.85$ , and  $m_4 = -7.92$ . Therefore, the transient component of the fault current has a general formulation of:  $i_{\text{transient}} = I_1 e^{m_1 t} + [Ae^{\lambda t} \cos(\mu t) + Be^{\lambda t} \sin(\mu t)] + I_2 e^{m_4 t}$ , where  $\lambda$  is the real and  $\mu$  is the complex part of the complex conjugate roots. Total fault current is then:  $i_f(t) = i_{\text{transient}} + i_{\text{sym}}$ .

Four unknown coefficients of the transient current,  $I_1$ ,  $A$ ,  $B$ , and  $I_2$  are determined from the initial currents of the circuit, including  $i_f(0^+) = 0$  and its three derivatives from the procedure described in Appendix II. This will result in  $I_1 = -158.25$ ,  $A = -3.52$ ,  $B = 58.60$ , and  $I_2 = 99.0$ . Now we have the complete fault current, which provides the maximum transient component at the start of the fault.

Three graphs are provided for analysis of the results. Fig. 7 shows the calculated total fault current, the calculated transient current, and the Thévenin (phasor-based) estimated dc component of the fault current. All values are in per unit of the amplitude of the steady state fault current ( $I_m$ ). Due to the decaying sinusoidal components in the calculated transient current, the phasor-based estimated dc current is fundamentally different, in nature and magnitude, from the actual transient current. While the phasor-based current is a simple decaying dc current, the calculated one is much more complex and takes a much longer time to diminish. The phasor-based current almost vanishes in three cycles after the fault, but the calculated transient current continues to grow to positive from its original negative value. No exponential dc current can simulate such a change from negative to positive, or vice versa. From the perspective of the circuit breaker interrupting rating, there is a major difference between the two currents. Between the third and fifth cycles after the fault, when breakers typically attempt to interrupt the current, there is practically no phasor-based dc

current while the calculated transient current is almost as large as the symmetrical fault current. This could subject a circuit breaker to a current beyond its interrupting capability.

Fig. 8 shows the absolute difference between the two transient currents, the actual calculated current and the phasor-based estimated one. Between the third and fifth cycles, this difference is as high as above 44%, which could have major consequences for a circuit breaker. Fig. 9 shows the individual components of the calculated transient current in per unit of the symmetrical fault current. While there are two decaying dc components, there are also two sinusoidal terms, the sum of which is shown as one current, as they have the same frequency. The graph shows how the dynamics of the transient components make the final total transient current.

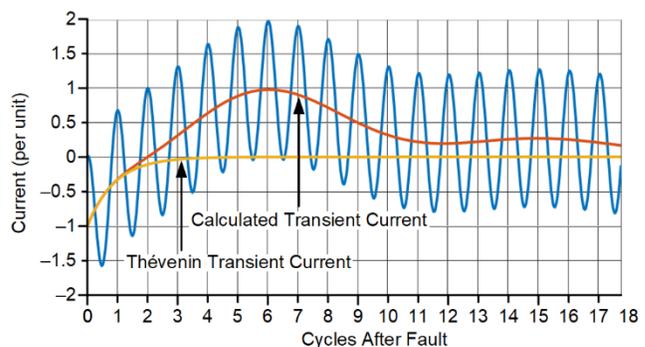


Fig. 7. Total Fault and Transient Currents—Example 2

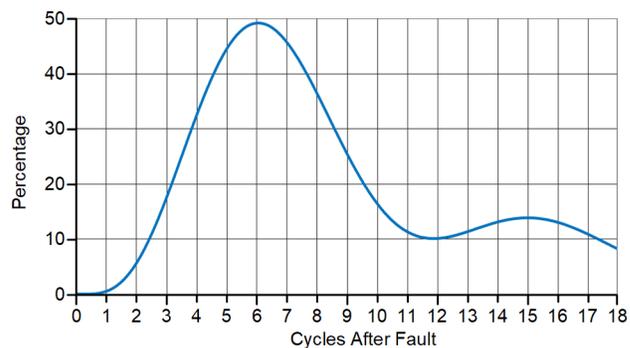


Fig. 8. Difference Between the Accurate and Phasor-Based Transient Currents—Example 2

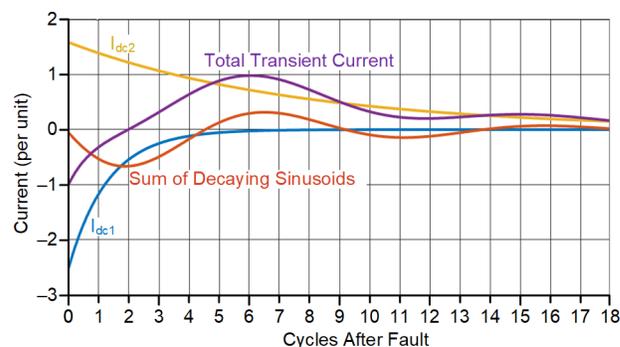


Fig. 9. Components of the Transient Current—Example 2

In general, for fourth and higher order circuits, where there are four or more parallel branches contributing to the fault, there is a high chance that the roots of the characteristic equation include complex conjugate pairs. Since complex

conjugate roots lead to decaying sinusoidal currents, as opposed to a simple decaying dc current, the estimated simplified phasor-based fault asymmetrical current could be substantially different from the actual fault current. The difference could potentially subject a circuit breaker to a much higher current at the time of interruption.

There are two major elements, one in circuit breaker sizing practices, and the other in manufacturing norms, that may help mitigate, and conceal, the problem of potential underestimating of transient fault currents. It is customary to apply a minimum of 25% margin to the interrupting capability of medium and high voltage circuit breakers. Per the IEEE simplified circuit breaker sizing method, if the steady state symmetrical fault current is below 80% of the breaker's interrupting capability, the breaker can be used at its full capacity without a concern about the system's X/R value. This amounts to the circuit breaker's rated short circuit current being 25% higher than the symmetrical fault current. On the other hand, high voltage circuit breakers are manufactured in only limited number of distinct interrupting capabilities. For the system voltages between 72.5 kV and 550 kV, the industry's predominant available short circuit ratings for circuit breakers are 31.5KA, 40 kA and 63 kA. The combined effect of these two practices is that there is often more than 25% margin in a circuit breaker interrupting capability, which could mask the impact of any overlooked transient fault current. However, the problem could resurface if the margin is below 25%, or in special scenarios such as in the proximity of power generation plants with multiple large generators, substations with 4 or more relatively strong incoming sources (Example 2), and substations with combined weak and strong sources (Example 1). The proposed method could be specially used in these cases to ensure sufficient capability of the circuit breakers.

The impact of the difference between the actual transient current and the phasor-based estimated one can be quantified based on the total asymmetrical *rms* current that a circuit breaker must interrupt. This will help us get a better insight into the real impact on circuit breakers and how we may simulate that with a variable X/R, as the breaker rating structure is based on symmetrical fault current and system's X/R. This is what we discuss in the next section.

## VI. THE CONCEPT OF A VARIABLE X/R

### A. Background

As we saw in the examples of the previous section, an estimated phasor-based dc current, with only one time-constant from the Thévenin equivalent of the circuit at the fault point, doesn't necessarily simulate the actual transient component of the fault current. This becomes particularly significant when the characteristic equation has complex conjugate roots, which creates decaying sinusoidal transient components. This makes the characteristics of the transient current quite different from a simple decaying dc current, as provided by the phasor-based method.

Circuit breakers are rated based on the *rms* value of the symmetrical fault current. To follow the *rms*-based rating practice, we need to calculate the *rms* value of the total asymmetrical fault current.

As a first step, let's compare the two total *rms* fault currents, one based on the actual calculation and the other based on phasor simplification. As described in Section III.A, the *rms* value of the total fault current from its symmetrical and transient components is:

$$I_{\text{asym}} = \sqrt{I_{\text{sym}}^2 + I_{\text{tr}}^2} \quad (7)$$

In the above equation,  $I_{\text{asym}}$  is the *rms* of the total fault current,  $I_{\text{sym}}$  is the *rms* of the symmetrical (steady state) fault current, and  $I_{\text{tr}}$  is the amplitude of the transient current.  $I_{\text{tr}}$  is generally a combination of decaying dc and sinusoidal components. On the other hand, the *rms* value of the asymmetrical fault current obtained from the phasor method is provided by (3). Assuming a Thévenin X/R, we get:

$$I_{\text{asym}_{\text{th}}} = I_{\text{sym}} \sqrt{1 + 2e^{\frac{-2\omega t}{X_{\text{th}}/R_{\text{th}}}}} \quad (8)$$

$I_{\text{asym}_{\text{th}}}$  is the Thévenin-based *rms* of the total fault current, while  $X_{\text{th}}$  and  $R_{\text{th}}$  are the Thévenin equivalent quantities at the fault point. The transient current associated with  $I_{\text{asym}_{\text{th}}}$  is a decaying dc current with only one time-constant, as we saw in Section III.A.

Fig. 10 and Fig. 11 show the two total *rms* fault currents, in per unit of the symmetrical *rms* current,  $I_{\text{sym}}$ , for Examples 1 and 2 in the previous section. As expected, *rms* currents based on actual calculations are larger than those based on phasor reduction (Thévenin) method. For the third order circuit of Example 1 and at the fourth cycle, the actual *rms* fault current is about 13% higher than the phasor-based calculation. For the same comparison in Example 2, we see that the actual *rms* current is 34% higher than the phasor-based *rms* current at the fourth cycle, which could be beyond the circuit breaker's interrupting capability. Both actual and phasor-based *rms* currents start at  $\sqrt{3}$  per unit, as can be verified from the way they are calculated.

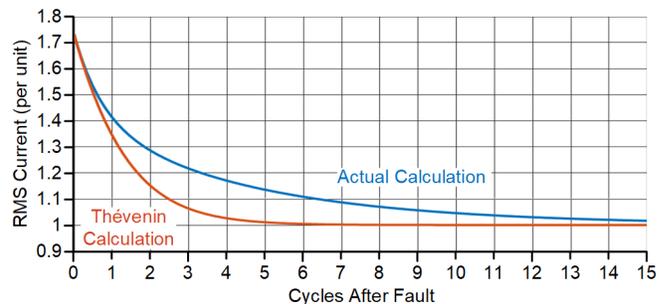


Fig. 10. Asymmetrical RMS Fault Currents—Example 1

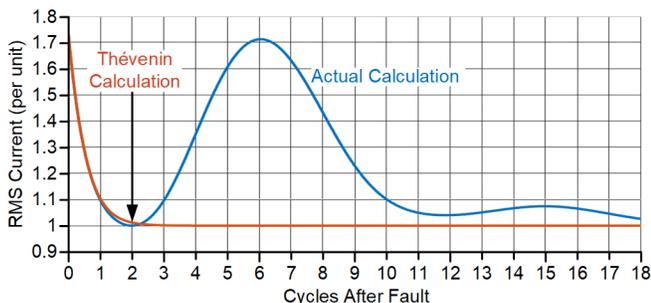


Fig. 11. Asymmetrical RMS Fault Currents—Example 2

### B. Defining a Variable $X/R$

Since the ODE-based calculation provides us with the accurate total fault current, we may use the actual total current and define a variable  $X/R$  in such a way that the asymmetrical fault current at any time is given as if the fault had started with that  $X/R$ . Then, instead of a constant  $X/R$  from the phasor method, we can use this actual  $X/R$  at any specific breaker's contact parting time for breaker sizing. This allows us to continue using the existing standards for circuit breaker sizing, as they are based on the concept of a single constant  $X/R$ ; only we would need to use different values at different contact parting times.

We can use (7) and assume that there is a time-dependent  $X/R$  that can be assigned to the total transient current. Calling the new varying quantity as  $\mathcal{X}/\mathcal{R}$ , we can use (8) to express the *rms* of the total asymmetrical current as:

$$I_{\text{asym}} = I_{\text{sym}} \sqrt{1 + 2e^{\frac{-2\omega t}{\mathcal{X}/\mathcal{R}}}} \quad (9)$$

In the above equation,  $I_{\text{asym}}$  is the actual total asymmetrical *rms* current, obtained from (7), in which  $I_{\text{tr}}$  is the ODE-based total transient current. Hence, everything is known in (9) except  $\mathcal{X}/\mathcal{R}$ , which can be calculated as:

$$\mathcal{X}/\mathcal{R} = \frac{-2\omega t}{\ln[0.5(k^2 - 1)]} \quad (10)$$

In this equation,  $k = \frac{I_{\text{asym}}}{I_{\text{sym}}}$ . This is a variable  $X/R$  that can

be used to come up with an equivalent  $X/R$  at any specific time, including at breaker parting time.

To get a positive  $\mathcal{X}/\mathcal{R}$ , the argument of the natural logarithm must be between zero and one; that is,  $0 < 0.5(k^2 - 1) < 1$ . This yields  $I_{\text{sym}} < I_{\text{asym}} < \sqrt{3}I_{\text{sym}}$ . We know that  $I_{\text{asym}}$  is greater than  $I_{\text{sym}}$ , so this part is automatically met. The above condition also requires that  $I_{\text{asym}} < \sqrt{3}I_{\text{sym}}$ . We saw that the maximum of  $I_{\text{asym}}$  occurs at the start of the fault as this is how we designed the calculation to have the maximum transient current at the beginning of the fault; this provides an asymmetrical *rms* value,  $I_{\text{asym}}$ , which is  $\sqrt{3}$  times the symmetrical *rms* current,  $I_{\text{sym}}$ . When all the components of the transient current are decaying dc currents, the second condition is also automatically met as the transient current cannot grow to become larger than the initial value. For the case of a decaying sinusoidal, typically we don't expect the total transient current to become greater than

its initial value, and hence don't expect the total *rms* current to become larger than  $\sqrt{3}$  times the symmetrical *rms* current. However, in some extreme scenarios, the total transient current may grow larger than its initial value in the opposite direction due to slow-decaying sinusoidal components, which would make  $\mathcal{X}/\mathcal{R}$  a negative number. Physically, this means that there is no  $X/R$  that can make the total current larger than its initial value, as we are modeling based on a single decaying dc current. In this situation, we may choose to limit the maximum  $\mathcal{X}/\mathcal{R}$  to a value equivalent to  $k$  just below  $\sqrt{3}$ .

We can now calculate  $\mathcal{X}/\mathcal{R}$  for the two examples of the previous section. Fig. 12 and Fig. 13 show  $\mathcal{X}/\mathcal{R}$  for each of the two examples. In Example 1,  $\mathcal{X}/\mathcal{R}$  starts from about 14, which is the same as the phasor-based  $X/R$ , and increases to about 30 at fourth cycle. So, for sizing a breaker with a contact parting time of 4 cycles we would need to use 30 instead of 14 for our  $X/R$ .

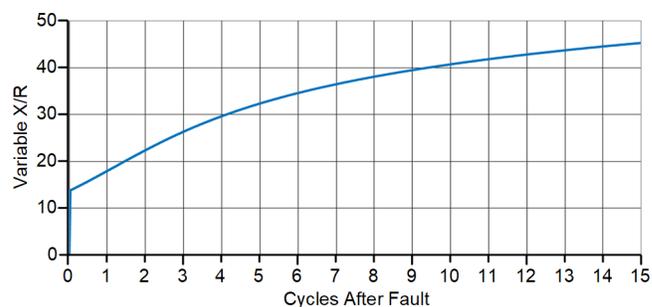


Fig. 12. Variable  $X/R$ —Example 1

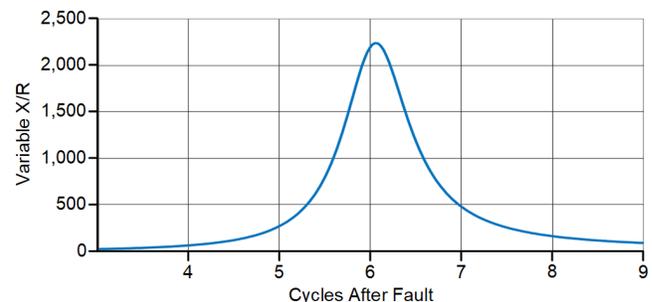


Fig. 13. Variable  $X/R$ —Example 2

For the plot of Example 2, the absolute value of the *rms* current increases, in the opposite direction, very close to its initial value at fault; hence,  $\mathcal{X}/\mathcal{R}$  jumps to very large numbers between fourth and ninth cycles. Physically, since the transient current goes back to absolute values close to its original value, a very large inductive circuit is needed to keep the current almost unchanged. This means that a circuit breaker will basically have to interrupt an *rms* current very close to the current at the start of the fault, with no decaying having occurred and with the transient component almost equal to the symmetrical component.

When  $\mathcal{X}/\mathcal{R}$  becomes too large to practically use in a breaker with a specified symmetrical rating, it's time to use a breaker with a higher symmetrical capability. For instance, the curves in C37.010 standard are limited to an  $X/R$  of 130. In the previous example, the transient component is close to its

original value at the start of the fault at the contact parting time, and this means that the breaker will have to interrupt an asymmetrical *rms* current of  $\sqrt{1+2(1)} = \sqrt{3}$  times the symmetrical fault current. So, if a new breaker with a symmetrical interrupting capability of  $\sqrt{3}$  times the original capability is selected, it would have enough capability.

In the case of very large calculated  $\mathcal{X}/\mathcal{R}$ , for example when it is greater than 130, we can calculate a new symmetrical rating for a circuit breaker based on a desired  $X/R$  at the breaker's contact parting time,  $t_c$ . Let's consider  $\frac{\mathcal{X}}{\mathcal{R}} = 17$  and come up with symmetrical rating of a circuit breaker, such that it will have the required asymmetrical interrupting capability obtained from (7) at the desired contact parting time. Equation (9), which is the same as (3), provides the required asymmetrical capability of a circuit breaker with a symmetrical rating of  $I_{\text{sym}}$ , provided that  $\frac{\mathcal{X}}{\mathcal{R}} \leq 17$ . Therefore, the symmetrical rating of a new circuit breaker is:

$$I_{\text{sym}_{\text{CB}}} = \frac{I_{\text{asym}_{\text{cal}}}}{\sqrt{1 + 2e^{-\frac{2\omega t_c}{17}}}} \quad (11)$$

$I_{\text{sym}_{\text{CB}}}$  is the calculated symmetrical rating of the circuit breaker based on the calculated total asymmetrical fault current,  $I_{\text{asym}_{\text{cal}}}$ , at the desired contact parting time of  $t_c$ , and assuming  $\frac{\mathcal{X}}{\mathcal{R}} = 17$ . In Example 2, where total asymmetrical fault current at fourth cycle is about 1.35 per unit of symmetrical fault current (from Fig. 11), the symmetrical rating of the circuit breaker from (11) is 95.1% of the calculated fault current at the 4<sup>th</sup> cycle, which is 1.283 times the steady state fault current in *rms* value. So, if we choose a circuit breaker with a symmetrical interrupting capability of 95.1% of the total fault current, the breaker can conveniently provide the desired operation. For contact parting time of 3 and 4 cycles, the symmetrical rating of a circuit breaker can be generally calculated from (11) per below, based on a calculated total fault current:

$$I_{\text{sym}_{\text{CB}}} = 0.951 I_{\text{asym}_{\text{cal}}}, t_c = 4 \text{ cycles}$$

$$I_{\text{sym}_{\text{CB}}} = 0.906 I_{\text{asym}_{\text{cal}}}, t_c = 3 \text{ cycles}$$

Note that the breaker's symmetrical rating calculated based on contact parting time of 3 cycles would be lower than when it is based on contact parting time of 4 cycles. This may seem counterintuitive. This is explained by considering that, for a circuit breaker to interrupt a current at 4<sup>th</sup> cycle, when the current has somewhat dropped from 3<sup>rd</sup> cycle, it must have a higher rating than when it is required to interrupt *the same* current at 3<sup>rd</sup> cycle.

## VII. CONCLUSION

The characteristics of fault currents in ac systems were reviewed and analyzed by solving the associated differential equation. We began with a simple RL circuit and observed how the transient current is a decaying dc current, whose time-

constant is defined by  $X/R$  of the system. The higher the  $X/R$ , the longer it takes for the transient current to diminish, hence a circuit breaker will have to interrupt a larger current.

The circuit breaker rating structure was reviewed, which is based on the steady state symmetrical fault current. A breaker rated at a certain symmetrical current is already capable of interrupting asymmetrical current if the system's  $X/R$  is not greater than 17. However, if it is greater than 17, the standard method is to adjust the symmetrical fault current by some multipliers and then compare that with a breaker rating to make sure there is enough interrupting capability.

In the standard sizing calculations of a circuit breaker it is assumed that the fault current follows the characteristic of a first order RL circuit: that it is fundamentally a decaying dc current and can be modeled by a single decaying exponential term with one time-constant. We saw that for second and higher order systems, where there are multiple parallel branches contributing to the fault, the transient component consists of more than one decaying term; and mathematically, the sum of two decaying dc currents cannot be expressed with an equivalent decaying term with only one time-constant, or one  $X/R$ . Therefore, using phasor-based Thévenin method to represent the system with only one  $X/R$  creates errors that could seriously interfere with circuit breaker sizing calculations. More importantly, we saw that, for higher order circuits, the transient current may not even be a sum of just simple decaying dc components; it may also consist of decaying sinusoidal terms. In this case, simulating the total transient current with one phasor-based  $X/R$  could create large errors in breaker sizing calculations and make a breaker undersized for the actual transient fault current.

To obtain the complete transient current when there are parallel circuits at the fault, we need to solve the ODE of the fault current. The complete solution of the fault current is the sum of the phasor-based steady state current and the transient current. To obtain the transient current, all we need is the characteristic equation of the fault current's ODE, when all the input voltages are ignored. Considering a defined circuit configuration as shown in Fig. 3, a general method was provided in Appendix I to calculate the coefficient of the characteristic equation for any  $n$ th order circuit. The roots of the characteristic equation provide the general formulation of the transient response. We then need to find the unknown coefficients of the general transient response. Appendix II provides details of how they can be calculated from the initial conditions of the system, which are currents and their derivatives.

The procedure to set up the fault current's ODE, solving for its general transient response, and finding the coefficient of the transient current are all algebraic operations that do not involve any direct method to solve a differential equation. The advantage of this method is that the complete procedure can be implemented by general purpose software, in algorithms that include only algebraic calculations.

Two sample systems, a third and a fourth order, were analyzed by the proposed method; the actual calculated fault current was compared to the phasor-based current which

showed large differences in both cases, particularly when there are decaying sinusoidal terms in the transient current.

Since circuit breakers are rated in *rms* current, the asymmetrical *rms* current was calculated in Section VI for both the actual and the phasor-based transient currents; this provides a measure of the real impact on a circuit breaker by the difference between the two transient currents. For the two examples, this resulted in *rms* currents that were 13% and 34% higher than the phasor-based currents at fourth cycle after the fault; this could exceed the interrupting capability of a circuit breaker.

Finally, to properly simulate the actual transient current calculated with one X/R, a variable X/R was defined that is dynamically linked to the actual *rms* fault current,  $I_{\text{asym}}$ , and provides a variable quantity,  $\mathcal{X}/\mathcal{R}$ . This varying  $\mathcal{X}/\mathcal{R}$  would provide the same asymmetrical *rms* fault current at any specific time, the same way that a first order circuit would have provided with a constant X/R. To size a circuit breaker, one would need to use this varying  $\mathcal{X}/\mathcal{R}$  at the breaker's contact parting time, instead of a constant X/R. When such a calculated  $\mathcal{X}/\mathcal{R}$  is too large for practical consideration, a circuit breaker's symmetrical rating can be calculated such that, based on an X/R of 17, it can conveniently interrupt the calculated total fault current at the desired contact parting time, as provided by (11).

### VIII. APPENDIX I: DERIVATION OF THE HOMOGENEOUS ODE

A general procedure is presented in this appendix to build the homogeneous ODE of the circuit in Fig. 3 for fault current,  $i_f$ . Although voltage sources and initial currents appear in the equations, they won't have any impact on the formulation of the homogeneous ODE; hence their details are not of concern as will be seen in this appendix.

Once the homogeneous ODE for first, second, and third order circuits are built, a pattern emerges for the general formulation of the  $n$ th order circuit, with  $n$  parallel branches, which provides the coefficients of the ODE. These coefficients are functions of the circuit's R and L elements only. With the coefficients of the fault current's ODE calculated, the characteristic equation can be solved for the general structure of the homogeneous response, which is the transient fault current.

For circuits of second order and higher we will have to deal with a linear system of differential equations. This would involve taking additional derivatives to eliminate all branch currents and express everything based on  $i_f$  only. This will quickly become too complicated to solve. However, by using the Laplace transform, the system of differential equations can be converted to a system of linear algebraic equations. The Laplace equation of the fault current can then be derived as shown in the example below for a third order circuit.

Consider the three parallel branches in Fig. 3 with the fault's resistance and inductance. Since the voltages of all parallel branches are equal, we can use this to build a system of

equations based on fault and branch currents. There are four unknown currents, but only three of them are independent, as the fourth one can always be calculated from the other three, as  $i_1 + i_2 + i_3 = i_f$  at all time. We choose one of the branch currents to be expressed by the other currents as we need  $i_f$  to be in the equations. An example of the time domain equation for the branch one and two voltages is:

$$R_1 i_1 + L_1 \frac{di_1}{dt} - R_2 i_2 - L_2 \frac{di_2}{dt} = v_1 - v_2$$

Applying the Laplace transform, we get the equation below. Note that the Laplace transform of a derivative is obtained from  $\mathcal{L}[f'(t)] = sF_s - f(0)$ , where  $f(0)$  is the initial value of the function.

$$R_1 I_1 + sL_1 I_1 - R_2 I_2 - sL_2 I_2 = V_1 - V_2 + L_1 i_1(0) - L_2 i_2(0)$$

In the above equation,  $I_1$ ,  $I_2$ ,  $V_1$ , and  $V_2$  are the Laplace-domain equivalents of  $i_1(t)$ ,  $i_2(t)$ ,  $v_1(t)$ , and  $v_2(t)$ , respectively, while  $i_1(0)$  and  $i_2(0)$  are the initial values of  $i_1(t)$  and  $i_2(t)$ ;  $s$  is the derivative operator,  $\frac{d}{dt}$ , in the Laplace domain. As will be seen in the next few paragraphs, the right side of this Laplace equation has no impact on formulation of the homogeneous ODE. So, for simplification it is shown as a function,  $\mathcal{F}(V_1, V_2)$ . This is a Laplace-domain function of  $V_1$  and  $V_2$ , which includes constants such as  $L_k i_k(0)$ . The equation can be rearranged for the currents as:

$$(R_1 + sL_1)I_1 - (R_2 + sL_2)I_2 = \mathcal{F}(V_1, V_2)$$

The same voltage equation can be formulated between the first branch and each of the rest of the branches, including the fault path. This creates  $n$  equations in total with a total of  $n$  unknown currents. One of the branch currents must be expressed in terms of all the other currents, making a total of  $n$  unknown currents, including the fault branch. For our example of the third order circuit, we may choose  $i_3(t)$  as the dependent current and write the voltage equation between branches one and three as below, knowing that  $i_3(t) = i_f(t) - i_1(t) - i_2(t)$ .

$$R_1 i_1 + L_1 \frac{di_1}{dt} - R_3 (i_f - i_1 - i_2) - L_3 \frac{d}{dt} (i_f - i_1 - i_2) = v_1 - v_3$$

Once all  $n$  voltage equations, corresponding to  $n$  branches and the fault branch, are similarly formulated and converted to the Laplace domain, a linear system of equations can be set up to calculate the fault current. This can then be converted back to the time domain for the homogeneous ODE of  $i_f(t)$ , from which the characteristic equation is obtained. For our example of third order circuit, the Laplace transform equations are shown below:

$$\begin{cases} (R_1 + sL_1)I_1 - (R_2 + sL_2)I_2 & = \mathcal{F}_1 \\ (R_1 + sL_1)I_1 + (R_f + sL_f)I_f & = \mathcal{F}_2 \\ [(R_1 + R_3) + s(L_1 + L_3)]I_1 + (R_3 + sL_3)I_2 - (R_3 + sL_3)I_f & = \mathcal{F}_3 \end{cases}$$

On the right side of the above equations are some functions of  $V_1$ ,  $V_2$ , and  $V_3$  that may also include the initial currents. Expressing this system of linear equations by matrix, we get:

$$\begin{bmatrix} (R_1+sL_1) & -(R_2+sL_2) & 0 \\ (R_1+sL_1) & 0 & (R_f+sL_f) \\ [(R_1+R_3)+s(L_1+L_3)] & (R_3+sL_3) & -(R_3+sL_3) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_f \end{bmatrix} = \begin{bmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \\ \mathcal{F}_3 \end{bmatrix} \quad (12)$$

If we call the matrix of the coefficients  $A$ , currents are obtained by multiplying both sides of the equation by the inverse of matrix  $A$ :

$$\begin{bmatrix} I_1 \\ I_2 \\ I_f \end{bmatrix} = A^{-1} \begin{bmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \\ \mathcal{F}_3 \end{bmatrix}$$

From linear algebra, the inverse of a matrix is calculated by dividing the *adjoint* of matrix  $A$  by the determinant of  $A$ , or

$$A^{-1} = \frac{\text{adj}(A)}{|A|}. \quad \text{The adjoint of a matrix is a matrix of cofactors}$$

of the transpose of  $A$ . We don't need to get into details of how adjoint of a matrix is calculated as it won't impact what we are looking for, which is the homogeneous ODE of the fault current; we only need to know that adjoint of a matrix has the same dimensions as the original matrix, whose elements are calculated by certain sums and products of the original matrix elements. For example, the adjoint of our matrix  $A$  will have the same number of row and columns, with elements that are some combinations of the various elements of  $A$ . Rewriting the previous equation, we have:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_f \end{bmatrix} = \frac{1}{|A|} \text{adj}(A) \begin{bmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \\ \mathcal{F}_3 \end{bmatrix}$$

In the above equation, the product of  $\text{adj}(A)$  and the matrix consisting of  $\mathcal{F}_1$ ,  $\mathcal{F}_2$ , and  $\mathcal{F}_3$  is a  $3 \times 1$  matrix, whose elements are still some functions of  $V_1$ ,  $V_2$ , and  $V_3$ , but different from their original formulation in that they have been multiplied by various combinations of products and sums of circuit elements,  $R$  and  $L$ , and then added up together. This includes derivatives of the voltage sources due to multiplications by  $s$  and higher orders of  $s$ . In short, the functions  $\mathcal{F}_1$ ,  $\mathcal{F}_2$ , and  $\mathcal{F}_3$  will change to some other functions of  $V_1$ ,  $V_2$ , and  $V_3$  but still some combinations of voltage sources and their derivatives, including multiplications by circuit elements. We can call the new Laplace functions  $\bar{\mathcal{F}}_1$ ,  $\bar{\mathcal{F}}_2$ , and  $\bar{\mathcal{F}}_3$ . In general, for an  $n$ th order circuit,  $\bar{\mathcal{F}}$  functions include voltage sources and their  $(n-1)$  derivatives. The matrix equation for currents can now be written as:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_f \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} \bar{\mathcal{F}}_1 \\ \bar{\mathcal{F}}_2 \\ \bar{\mathcal{F}}_3 \end{bmatrix}$$

Fault current in the Laplace domain,  $I_f$ , can now be obtained from the above equation as:  $|A|I_f = \bar{\mathcal{F}}_3$ . The homogeneous ODE is obtained by making the forcing function zero. Therefore, the homogeneous ODE of the fault current in the Laplace domain is:

$$|A|I_f(s) = 0 \quad (13)$$

Matrix  $A$  is the matrix of the coefficients in (12) and  $|A|$  is the determinant of  $A$ . Since  $A$  is always some combination of products and sums of circuit elements, plus the operator  $s$ ,  $|A|$  will consist of products and sums of circuit elements as well as  $s$  operator. It can be shown that for an  $n$ th order circuit,  $|A|$  will have  $s$  operator and its  $n$  powers,  $s$ ,  $s^2$ , ...,  $s^n$ . Since  $s^n$  is equivalent to the  $n$ th derivative in time-domain, this means that the circuit in Fig. 3 with  $n$  parallel circuits will have an  $n$ th order ODE for the fault current.

Equation (13) is valid for any number of parallel branches in Fig. 3, provided that the corresponding  $|A|$  is calculated. If  $|A|$  is rearranged in a descending order of power of  $s$ , from  $n$ th power to zeroth power of  $s$ , it will have a general formulation as:

$$|A| = k_n s^n + k_{(n-1)} s^{(n-1)} + \dots + k_1 s + k_0 \quad (14)$$

By setting (14) equal to zero, we have in fact set up the characteristic equation of the homogeneous ODE, as  $s^n$  is equivalent to the  $n$ th derivative,  $\frac{d^n}{dt^n}$ , in the time domain and  $k$  coefficients are simple constants. So, all we need to do is to set up the matrix of coefficients,  $A$ , and then calculate the determinant of  $A$  to get to the characteristic equation.

For our example of a third order circuit, coefficients of (14) are per below, which are the coefficients of the third order polynomial characteristic equation:

$$k_3 = L_1 L_2 L_3 + L_1 L_2 L_f + L_1 L_3 L_f + L_2 L_3 L_f$$

$$k_2 = L_1 L_2 R_3 + L_1 L_2 R_f + L_1 L_3 R_2 + L_1 L_3 R_f + L_1 L_f R_2 + L_1 L_f R_3 + L_2 L_3 R_1 + L_2 L_3 R_f + L_2 L_f R_1 + L_2 L_f R_3 + L_3 L_f R_1 + L_3 L_f R_2$$

$$k_1 = L_1 R_2 R_3 + L_1 R_2 R_f + L_1 R_3 R_f + L_2 R_1 R_3 + L_2 R_1 R_f + L_2 R_3 R_f + L_3 R_1 R_2 + L_3 R_1 R_f + L_3 R_2 R_f + L_f R_1 R_2 + L_f R_1 R_3 + L_f R_2 R_3$$

$$k_0 = R_1 R_2 R_3 + R_1 R_2 R_f + R_1 R_3 R_f + R_2 R_3 R_f$$

For the circuit arrangement of Fig. 3, the matrix of coefficients,  $A$ , has an easy pattern to identify for a general  $n$ th order circuit, which consists of  $n$  parallel branches. It is an  $n \times n$  matrix with the following structure:

- The first  $(n-2)$  rows: The first column is always  $(R_1 + sL_1)$ . Then, for the  $k$ th row ( $k \leq (n-2)$ ), there will be  $-(R_{k+1} + sL_{k+1})$  on the  $(k+1)$ th column. The rest are zero.
- On row  $(n-1)$ , which belongs to the fault path, the first column is similarly  $(R_1 + sL_1)$ , and on column  $n$  there will be  $(R_f + sL_f)$ . The rest are zero.
- Finally, on row  $n$ , which is for the  $n$ th parallel branch whose current is expressed versus all other currents in the circuit, there will be  $[(R_1 + R_n) + s(L_1 + L_n)]$  on the first column,  $R_n + sL_n$  on columns 2 to  $(n-1)$ , and  $-(R_n + sL_n)$  on the last column,  $n$ .

Hence, matrix  $A$  can be constructed with no need to write down any electrical equations, provided that the circuit is arranged per Fig. 3 and the current of the last parallel branch is expressed versus all the other currents in the circuit.

Once matrix  $A$  is set up, its determinant provides the characteristic equation of the homogeneous ODE for the fault current. Software packages can be used to calculate the determinant of matrix  $A$ , which includes operator  $s$  per (14). The roots of the characteristic equations will then provide the general transient current whose unknown coefficients need to be calculated based on the initial conditions of the circuit, as described in Appendix II.

## IX. APPENDIX II: CALCULATING THE COEFFICIENTS OF THE TRANSIENT CURRENT

The characteristic equation obtained in Appendix I provides the general response of the homogeneous equation, which is the general formulation of the transient fault current. We saw that, depending on the type of the roots of the characteristic equation, the transient current has a general formulation of  $\sum I_k e^{\alpha_k t} + \sum [A_i e^{\lambda_i t} \cos(\mu_i t) + B_i e^{\lambda_i t} \sin(\mu_i t)]$ , where  $\alpha_k$  is a real root and  $\lambda_i \pm j\mu_i$  is complex root as a complex conjugate pair.  $I_k$ ,  $A_i$ , and  $B_i$  are unknown coefficients that need to be determined based on the initial conditions of the circuit at the time of fault. For an  $n$ th order circuit, there are  $n$  roots for the characteristic equations, and hence  $n$  coefficients to be determined, one per each root. This requires  $n$  equations to be solved for all the coefficients of the transient response.

One obvious equation is  $i_f(0) = 0$ , as the continuity principle in inductors doesn't allow for an instantaneous change in current. The other  $(n-1)$  equations come from the  $(n-1)$  derivatives of  $i_f$  at  $t = 0$ . The time that the derivatives are going to be calculated will be shown as  $0^+$  to highlight the fact that the derivatives are calculated instantly *after* the fault. So, the task is to calculate the  $(n-1)$  derivatives of  $i_f$  at  $t = 0^+$ .

In Fig. 3, at the instant of the switch being closed, we have the following KVL equation for a loop consisting of each parallel branch and the fault branch, for  $1 \leq k \leq n$ :

$$R_k i_k(0^+) + L_k \frac{di_k(0^+)}{dt} + R_f i_f(0^+) + L_f \frac{di_f(0^+)}{dt} = v_k(0^+)$$

Since  $i_k(0^+)$  is known from the pre-fault phasor calculations, the above equation can be rearranged to have all the unknowns on the left side and the known quantities on the right side:

$$L_k \frac{di_k(0^+)}{dt} + R_f i_f(0^+) + L_f \frac{di_f(0^+)}{dt} = v_k(0^+) - R_k i_k(0^+)$$

Although  $i_f(0^+)$  is zero, we need to keep it as we will be taking derivatives of this equation. It is important to realize that although, due to the current continuity principle,  $i_k(0^+)$  is calculated from the pre-fault currents, its derivative,  $\frac{di_k(0^+)}{dt}$ ,

cannot be similarly calculated. Although branch currents don't instantaneously change when the switch is closed, the rate of change of the currents will instantly change. This is because the

fault path is now added to the circuit and  $\frac{di_k(0^+)}{dt}$  must follow

the above KVL equation. From a physical perspective,  $\frac{di_k(0^+)}{dt}$  determines the inductor voltages and there is no continuity requirement for the voltage of an inductor; in fact, the voltages across  $L_k$  and  $L_f$  do change at the instant of the fault.

The above KVL equation provides  $n$  equations, corresponding to each parallel branch. We have total of  $(n+1)$  unknowns, including  $\frac{di_k(0^+)}{dt}$  and  $\frac{di_f(0^+)}{dt}$ , so we need another equation to solve for the unknowns. The last equation comes from  $\sum i_k(0^+) = i_f(0^+)$ , which provides:

$$\left( \sum \frac{di_k(0^+)}{dt} \right) - \frac{di_f(0^+)}{dt} = 0.$$

We now have  $(n+1)$  linear equations for  $\frac{di_k(0^+)}{dt}$  and  $\frac{di_f(0^+)}{dt}$ , per below, from which all the first derivatives of currents are calculated:

$$\begin{cases} L_k \frac{di_k(0^+)}{dt} + R_f i_f(0^+) + L_f \frac{di_f(0^+)}{dt} = v_k(0^+) - R_k i_k(0^+) \\ \frac{di_1(0^+)}{dt} + \frac{di_2(0^+)}{dt} + \dots + \frac{di_n(0^+)}{dt} - \frac{di_f(0^+)}{dt} = 0 \end{cases} \quad (15)$$

This is a linear system of equations with constant coefficients that can be set up and solved by computer algorithms.

So far, we have calculated  $i_f$  and its first derivative, which is good enough if we have a second order circuit with two unknown coefficients in the transient current. Similar calculations can be performed for higher order circuits with a higher number of unknown coefficients in the transient current. We then need to take one derivative of (15) for each additional order. In general, for an  $n$ th order circuit, we need to take  $(n-1)$  derivatives of (15). This will provide  $i_f$  and its  $(n-1)$  derivatives that can be used to find the coefficients of the transient fault current. For example, for a third order circuit, we need to take one more derivative of (15) per below:

$$\begin{cases} L_k \frac{d^2 i_k(0^+)}{dt^2} + R_f \frac{di_f(0^+)}{dt} + L_f \frac{d^2 i_f(0^+)}{dt^2} = \frac{dv_k(0^+)}{dt} - R_k \frac{di_k(0^+)}{dt} \\ \frac{d^2 i_1(0^+)}{dt^2} + \frac{d^2 i_2(0^+)}{dt^2} + \dots + \frac{d^2 i_n(0^+)}{dt^2} - \frac{d^2 i_f(0^+)}{dt^2} = 0 \end{cases}$$

Solving this system of linear equations, with  $1 \leq k \leq n$ , will provide  $\frac{d^2 i_k(0^+)}{dt^2}$  and  $\frac{d^2 i_f(0^+)}{dt^2}$ . Note that in the above equations,  $\frac{di_k(0^+)}{dt}$  and  $\frac{di_f(0^+)}{dt}$  are known quantities obtained from the previous system of equations that was solved one step before; and  $\frac{dv_k(0^+)}{dt}$  is directly obtained from each voltage source function.

Once all  $(n-1)$  derivatives of  $i_f(0^+)$  are calculated, we will have  $n$  equations, including  $i_f(0^+) = 0$ , to solve for the unknown coefficients of the transient current. It is very important to keep in mind that when substituting  $i_f(0^+)$  and its derivatives to calculate the coefficients, they must be substituted in the complete fault current,  $i_f(t) = I_{\text{transient}} + I_m \cos(\omega t + \theta)$ , not in the transient current. It is also important to recall that although  $t = 0$  is the reference time for the fault, for any calculation on the pre-fault quantities such as  $i_k(0)$ ,  $v_k(0)$  and derivatives of the voltages, the fault occurs at  $t = t_0$ , where  $t_0 = \frac{-\theta}{\omega}$  or  $t_0 = \frac{\pi - \theta}{\omega}$  as we saw in Section IV.D.

All these calculations are repetitive algebraic operations that can be computerized for any circuit order. Once the coefficients of the transient current are calculated as described here, the complete maximum fault current is obtained by adding the steady state current, from phasor calculations, to the transient current, obtained from solving the homogeneous ODE of the fault current.

## X. REFERENCES

- [1] IEEE C37.010-1999 (R2005), IEEE Application Guide for AC High-Voltage Circuit Breakers Rated on a Symmetrical Current Basis.
- [2] IEEE C37.04-1999 (R2006), IEEE Standard Rating Structure for AC High-Voltage Circuit Breakers.
- [3] AIEE Committee Report, "Calculation of Electric Power System Short Circuits During the First Few Cycles," *Transactions of the American Institute of Electrical Engineers*, Vol. 75, Issue 3-III, 1956, pp. 120-127.

- [4] AIEE Committee Report, "A New Basis for Rating Power Circuit Breakers," *Transactions of the American Institute of Electrical Engineers*, Vol. 73, Issue 1-III-A, 1954, pp. 353-367.
- [5] W. C. Huening, Jr., "Interpretation of New American National Standards for Power Circuit Breaker Applications," *IEEE Transactions on Industry and General Applications*, Vol. IGA-5, Issue 5, September 1969, pp. 501-523.
- [6] W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*. 10th ed. John Wiley & Sons, Inc., Hoboken, NJ, 2012.
- [7] C. K. Alexander and M. N. O. Sadiku, *Fundamentals of Electric Circuits*. 5th ed. McGraw-Hill, New York, NY, 2013.

## XI. BIOGRAPHY

**Amir Norouzi** has been with Schweitzer Engineering Laboratories, Inc. (SEL) from 2017 as a protection application engineer. He received a BSc degree from Tehran Polytechnic in 1996 and an MSc degree from the University of New Brunswick, NB, Canada in 2003, both in electrical engineering. He has 19 years of experience in power system engineering, including generation, transmission, industrial and renewables. Prior to joining SEL, he was with General Electric for 6 years in various engineering roles in power system protection and control.